Solution to Assignment 9

Section 9.1:

(8). Take $a_n = \dfrac{(-1)^n}{\sqrt{n}}$. You may use definition to show it converges, but later you can use the Alternating Test.

(9). For $\varepsilon > 0$, there is some n_0 such that

$$|\sum_{k=m}^{n} a_k| < \varepsilon/2 \ , \quad \forall m, n \geq n_0 \ .$$

But then

$$na_n = (n - n_0)a_n + n_0 a_n \leq a_{n_0} + \cdots + a_n + n_0 a_n < \varepsilon/2 + n_0 a_n \ .$$

As $\sum a_n$ converges implies $\lim_{n\to\infty} a_n = 0$, we can find some $n_1 \geq n_0$ such that $n_0 a_n < \varepsilon/2$ for all $n \geq n_1$. Putting things together, for $n \geq n_1$,

$$0 \leq na_n < \frac{\varepsilon}{2} + n_0 a_n < 2 \times \frac{\varepsilon}{2} = \varepsilon \ .$$

(10). Take $a_n = 1/(n \log n)$, $n \geq 2$, by the Integral Test or other means, you get the desired result.

(11). The assumption implies that there is some α and n_0 such that $|n^2 a_n - \alpha| \leq 1$ for all $n \geq n_0$. Therefore,

$$\left|\sum_{k=m}^{n} a_k\right| \leq (|\alpha| + 1) \sum_{k=m}^{n} \frac{1}{k^2} \ , \quad n, m \geq n_0 \ .$$

As $\sum n^{-2} < \infty$, for $\varepsilon > 0$, there is some $n_1 \geq n_0$ such that $\sum_{k=m}^{n} k^{-2} < \varepsilon/(|\alpha| + 1)$, so $|\sum_{k=m}^{n} a_k| < \varepsilon$ for all $m, n \geq n_1$ too.

(13a).

$$\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} = \frac{1}{(\sqrt{n+1} + \sqrt{n})\sqrt{n}} \geq \frac{1}{2(n+1)} \ .$$

As $\sum 1/(n+1)$ is divergent, this series is also divergent.

(13b).

$$\frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{(\sqrt{n+1} + \sqrt{n})n} \leq \frac{1}{n^{3/2}} \ .$$

As $\sum n^{-3/2} < \infty$, this series is absolutely convergent.

Section 9.2

1. (a) Let $x_n := \frac{1}{(n+1)(n+2)}$. We have $|x_{n+1}/x_n| = (n+1)/(n+3) = 1 - 2/(n+3)$, so

$$\lim_{n\to\infty} n\left(1 - \frac{|x_{n+1}|}{|x_n|}\right) = 2 > 1 \ .$$

By the limit version of Raabe's Test, the series converges absolutely.

An alternate method. Observe that $x_n \le 1/n^2$ and $\sum_n n^{-2} < \infty$. By Comparison test, $\{\sum x_n\}$ converges absolutely since each x_n is positive.

(c) Since $\lim_{n\to\infty} 2^{-1/n} = 2^0 = 1 \neq 0$. $\{\sum 2^{-1/n}\}$ diverges.

2. (b) Observing that

$$\frac{1}{(n^2(n+1))^{1/2}} \le \frac{1}{n^{3/2}} \ ,$$

and the fact that $\sum 1/n^{3/2} < \infty$, we conclude by the Comparison Test that this series is absolutely convergent.

(c) Since

$$\frac{|x_{n+1}|}{|x_n|} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \left(1 + \frac{1}{n}\right)^{-n} \to e^{-1} < 1 \ ,$$

as $n \to \infty$, we apply the limit version of Ratio Test to conclude absolute convergence.

An alternate method. $x_n = \dfrac{n!}{n^n} = \dfrac{n(n-1)\cdots 21}{nn\cdots nn} \le \dfrac{2}{n^2}$. Therefore by comparison test, the series diverges.

(d) Denote $\{x_n := (-1)^n \frac{n}{n+1}\}$. Then $\{\lim x_{2n} = 1\}$ and $\{\lim x_{2n-1} = -1\}$. Since there is no limit (let alone tending to 0), $\{\sum x_n\}$ diverges. Alternatively, you may argue by $\lim_{n\to\infty} x_n$ does not tend to 0.

3. (b) We have

$$((\log n)^{-n})^{1/n} = 1/\log n \to 0$$

as $n \to \infty$, by the limit version of Root Test we conclude that the convergence is absolute.

(c) See if we can find some n_0 such that

$$(\log n)^{-\log n} \le n^{-2}$$

for $n \ge n_0$. Taking log both sides to get

$$(-\log n)\log\log n \le -2\log n \ ,$$

which is

$$-\log\log n \le -2,$$

and it holds for some n_0. Hence the series is convergent by Comparison Test.

(d) Using $\log n \le n$, we have

$$\frac{1}{(\log n)^{\log\log n/n}} \ge \frac{1}{n^{\log\log n/n}} \ .$$

Using $\log\log n \le \log n \le n$ we further have

$$\frac{1}{n^{\log\log n/n}} \ge \frac{1}{n^{n/n}} = \frac{1}{n} \ .$$

As $\sum_n n^{-1} = \infty$, by comparison test

$$\sum_n \frac{1}{(\log n)^{\log\log n/n}} \ge \sum_n \frac{1}{n} = \infty \ .$$

That is, this series is divergent.

(e) Use Integral Test to the function $f(x) = \log\log x$ to conclude divergence.

4. (b) Denote $\{x_n := n^n e^{-n}\}$. We have $|x_n|^{1/n} = n/e \to \infty$ as $n \to \infty$. By the limit version of Root Test we have divergence. You may also use the limit version of Ratio Test.

(c) $a_n = e^{-\log n} = 1/n$ is divergent.

(d) We use Ratio Test. We have

$$\frac{|x_{n+1}|}{|x_n|} = \frac{\log(n+1)}{\log n}\frac{1}{e^{\sqrt{n}+\sqrt{n+1}}} \to 0 \ ,$$

as $n \to \infty$. By the limit version of Ratio Test, this series is absolutely convergent.

(e) $a_n = n!e^{-n}$. By Ratio Test in Limit Form, $a_{n+1}/a_n = e/(n+1) \to 0$, hence it is convergent.

6. A routine application of the Integral Test after letting $\{f(x) := (ax+b)^{-p}\}$. Or reduce it to the standard case $\sum 1/n^p$ by applying the Comparison Test in view of

$$\frac{1}{(a+b)^p n^p} \le \frac{1}{(an+b)^p} \le \frac{1}{n^p} \ .$$

7. (a) Denote $\{x_n := \frac{n!}{3\cdot 5\cdot 7\cdots(2n+1)}\}$. Then $\{\left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{2n+3} \to \frac{1}{2} < 1\}$.
By the limit form of Ratio Test, $\{\sum x_n\}$ converges absolutely.

(b) $a_n = (n!)^2/(2n)!$. As $a_{n+1}/a_n = (n+1)^2/(2n+1)(2n+2) \to 1/4$, it is convergent by Ratio Test (Limit Form).

(c) Denote $\{x_n := \frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}\}$. Then $\{\left|\frac{x_{n+1}}{x_n}\right| = \frac{2n+2}{2n+3} = 1 - \frac{1}{2n+3}.\}$ Therefore,

$$\lim_{n\to\infty} n\left(1 - \left(1 - \frac{1}{2n+3}\right)\right) = \frac{1}{2} \ ,$$

which implies that the series diverges by the limit form of Raabe's Test.

8. Note that this series is a rearrangement of $a, a^2, \dots, a^{n-1}, a^n, \dots$, which we already know is absolutely convergent.
Root test:
$$|x_n|^{1/n} = \begin{cases} a^{(n-1)/n}, & \text{n = 2k;} \\ a^{n/(n-1)}, & \text{n = 2k-1.} \end{cases}$$
In both cases $|x_n|^{1/n} < 1$. By root test, the infinite series is convergent.
Ratio test:
$$\frac{x_{n+1}}{x_n} = 1/a > 1 \quad \forall n = 2k+1, k \in \mathbb{N}$$
and
$$\frac{x_{n+1}}{x_n} = a^2 < 1 \quad \forall n = 2k, k \in \mathbb{N}$$
We can't use ratio test to judge if this series is convergent.

17. Applying the limit version of Raabe's Test
$$n\left(1 - \frac{x_{n+1}}{x_n}\right) = \frac{n(q-p)}{q+n+1} \to q - p \ , \qquad \text{as } n \to \infty \ .$$
Therefore, we have convergence if $q - p > 1$ and divergence if $q - p < 1$. When $q = p + 1$, $\sum_n x_n = \sum_n 1/(q+n) = \infty$, so we have divergence in this case.

19. We adopt the notation in the question. Since $b_1 = \sqrt{A} - \sqrt{A_1}$ and $b_n = \sqrt{A - A_{n-1}} - \sqrt{A - A_n} > 0$,
$$\sum_{k=1}^{N} b_k = \sqrt{A} - \sqrt{A - A_N} \to \sqrt{A} \text{ as } N \to \infty.$$
Hence the series converges. Now, let us verify that $\lim_{n\to\infty} a_n/b_n = 0$. For $n > 1$,
$$b_n = \sqrt{A - A_{n-1}} - \sqrt{A - A_n} = \frac{A_n - A_{n-1}}{\sqrt{A - A_{n-1}} + \sqrt{A - A_n}} = \frac{a_n}{\sqrt{A - A_{n-1}} + \sqrt{A - A_n}}.$$
Using the fact that $\lim_{n\to\infty} A_n = A$, we conclude that
$$\frac{a_n}{b_n} = \sqrt{A - A_{n-1}} + \sqrt{A - A_n} \to 0 \text{ as } n \to \infty.$$

20. Let $b_n = a_n/\sqrt{A_n}$ where A_n is the nth partial sum of $\sum a_n$. It is clear that
$$\lim_n (b_n/a_n) = \lim_n 1/\sqrt{A_n} = 0$$
since $\sum a_n$ is divergent. Now we prove $\sum b_n$ is also divergent.
$$\sum b_n \geq \sum_{n=1}^{M} b_n \geq \sum_{n=1}^{M} a_n/\sqrt{A_M} = \sqrt{A_M} \quad \forall M \in \mathbb{N}$$
Letting $M \to \infty$, we have the desired conclusion.

Supplementary Exercises

1. Consider $\sum_{n=1}^{\infty} a_n$ and let $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ where $b_n = a_n^+$ and $c_n = a_n^-$ (so $a_n = a_n^+ - a_n^-$). Show that $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both are divergent to infinity when $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

   Solution. In case one of these series is convergent, say $\sum b_n$, let us show that $\sum c_n$ is also convergent, so $\sum |a_n| = \sum b_n + \sum c_n$ is also convergent, contradicting that $\sum a_n$ is only conditionally convergent. Let $\varepsilon > 0$, there is some n_0 such that $|a_{m+1} + \cdots + a_n| < \varepsilon/2$ for all $n, m \geq n_0$. On the other hand, choose $n_1 \geq n_0$, $b_{m+1} + \cdots + b_n < \varepsilon/2$ for all $n, m \geq n_1$. By subtracting these two inequalities and by choosing indices properly, we have $c_{m+1} + \cdots + c_n < \varepsilon$ for all $n, m \geq n_1$, done.

2. Show that every conditionally convergent series admits a rearrangement which is divergent to infinity.

   Solution. Adapting the notation in the previous problem, first we pick $b_1, \cdots, b_{n_1}$ such that $b_1 + \cdots + b_{n_1} > 1 + c_1$. Next, add $-c_1$ to the finite sequence to get $\{b_1, b_2, \cdots, b_{n_1}, -c_1\}$. Then add $b_{n_1+1}, \cdots, b_{n_2}$ so that $b_1 + b_2 + \cdots + b_{n_1} - c_1 + b_{n_1+1} + \cdots + b_{n_2} > 2 + c_2$. Add $-c_2$ to get $\{b_1, b_2, \cdots, b_{n_1}, -c_1, b_{n_1+1}, \cdots, b_{n_2}, -c_2\}$. Then add $b_{n_2+1}, \cdots, b_{n_3}$ so that $b_1 + \cdots - c_2 + b_{n_2+1} + \cdots + b_{n_3} > 3 + c_3$. By repeating the construction, we obtain a rearrangement whose partial sum is greater than any n. Note that this is possible because $\sum b_n = \infty$.

   Note. A theorem of Riemann states that given any number s including $\pm\infty$, there is a rearrangement on a conditionally convergent series converging to this number. You may google for it.